Solutions Set: Group Theory Exam October 2020

1. Let S_8 be the permutation group on $\{1,2,3,4,5,6,7,8\}$.

   (a) [**2 points**] Compute the order of $(1243)(4536)(78) \in S_8$.

   Solution: We first write the permutation as a product of disjoint cycles: $(1243)(4536)(78) = (36)(4512)(78)$. Then by Theorem IV.2.8, we have $\mathrm{ord}((36)(4512)(78)) = \mathrm{lcm}(2,4,2) = 4$.

   (b) [**3 points**] Let $\sigma_1 = (134)$ and $\sigma_2 = (278)$ be elements in S_8. Find an element $\tau \in S_8$ such that $\sigma_1 = \tau\sigma_2\tau^{-1}$.

   Solution: By Lemma IV.3.6, we have $\tau(278)\tau^{-1} = (\tau(2)\tau(7)\tau(8)) = (134)$. So $\tau(2) = 1$, $\tau(7) = 3$, $\tau(8) = 4$ and one can take $\tau = (84)(73)(12)$. (One can also compute τ directly.)

   (c) [**2 points**] Show that the elements (1354672) and (18967543) in S_8 are not conjugate.

   Solution: Suppose that they are conjugate. Then there exists $\tau \in S_8$ such that $(1354672) = \tau(18967543)\tau^{-1}$. The left hand side is an even permutation and the right hand side is an odd permutation. So we get a contradiction.
   Alternative solution: Suppose that they are conjugate. Then there exists $\tau \in S_8$ such that $(1354672) = \tau(18967543)\tau^{-1}$. Since the conjugation map $\gamma_\tau : S_8 \to S_8$ is an isomorphism, the orders of $\gamma((18967543)) = (1354672)$ and (18967543) must be the same. However,

   $$\mathrm{ord}((18967543)) = 8 \neq 7 = \mathrm{ord}((1354672)).$$

   So we get a contradiction.

   (d) [**3 points**] Describe an abelian subgroup of S_8 with 15 elements.

   Solution: The cyclic group generated by $(123)(45678)$ is abelian and has 15 elements since $\mathrm{lcm}(3,5) = 15$.

2. Let H, K be subgroups of a finite group G and $HK := \{hk \mid h \in H, k \in K\}$.

   (a) [**3 points**] Using $HK = \bigcup_{h\in H} hK$, show that $|HK| = |H||K|$ when $H \cap K = \{e\}$.

   Solution: By Lemma VII.1.7, we have $h_1K = h_2K \iff h_1h_2^{-1} \in K$. Since $H \cap K = \{e\}$ we get $h_1h_2^{-1} = e$, i.e., $h_1 = h_2$. So $h_1K \neq h_2K$ if $h_1 \neq h_2$. This implies $\#\{hK : h \in H\} = |H|$. Combining this with the fact that left cosets of K are pairwise disjoint (see the proof of Lagrange's theorem), we get

   $$|HK| = |\bigcup_{h\in H} hK| = \sum_{h\in H} |hK|.$$

   Since $|gK| = |K|$ for all $g \in G$, we can conclude $\sum_{h\in H} |hK| = |H||K|$.

   (b) [**3 points**] Show that if K is a normal subgroup then HK is a subgroup of G.

   We will check $H1, H2$ and $H3$:

   (H1) Since H, K are subgroups in G, we have $e \in HK$.

   (H2) If h_1k_1 and h_2k_2 are in HK then

   $$h_1k_1h_2k_2 = h_1(h_2h_2^{-1})k_1h_2k_2 = h_1h_2(h_2^{-1}k_1h_2)k_2 \in HK$$

   since $h_2^{-1}k_1h_2 \in K$.

   (H3) If $hk \in HK$ then $k^{-1}h^{-1} = h^{-1}hk^{-1}h^{-1} \in HK$ since $hk^{-1}h^{-1} \in K$.

(c) [**3 points**] Let G be a subgroup of S_5. Suppose $(13425) \in G$, and suppose G contains a subgroup isomorphic to S_4. Show that $G = S_5$.

> Solution: Let K denote the subgroup in S_5 isomorphic to S_4 and let $H := \langle(13425)\rangle$. We have $|K| = 24$ and $|H| = 5$. Since every element in H has order 5 and since $5 \nmid 24$, we have $K \cap H = \{(1)\}$. So by (a) we have $|HK| = |H||K| = 120$. Since we have $HK \subset G \subset S_5$ and $|HK| = |S_5|$, we get $G = HK = S_5$. (You can also solve this without using (a).)

3. [**4 points**] Let G be a group of order 105 and H be a group of order 28. Suppose $\varphi : G \to H$ is a homomorphism, and assume that there exists an element $a \in G$ such that $\varphi(a) \neq e_H$. Find the cardinalities $|\varphi(G)|$ and $|\ker\varphi|$.

> Solution: Existence of an element $a \in G$ with $\varphi(a) \neq e_H$ implies φ is not the trivial homomorphism. So $|\varphi(G)| \neq 1$. Since $\varphi(G)$ is a subgroup in H and hence by Lagrange's theorem, we have $|\varphi(G)| \in \{2, 4, 7, 14, 28\}$. On the other hand, by the homomorphism theorem we have
> $$G/\ker(\varphi) \cong \varphi(G)$$
> and since $\ker(\varphi) \leq G$ then by Lagrange's theorem we get $|\varphi(G)| = |G/\ker(\varphi)| \in \{3, 5, 7, 15, 21, 35, 105\}$. The only integer in the intersection is 7 so $|\varphi(G)| = 7$ and hence $|\ker(\varphi)| = 15$.

4. [**4 points**] Up to isomorphism, describe all abelian groups of cardinality 200 containing an element of order 50.

> Solution: In the question it is asked to describe abelian groups so we can use the structure theorem. By the structure theorem, up to isomorphism, the abelian groups of order 200 are
> $$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$$
> $$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$$
> $$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$$
> $$\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/20\mathbb{Z}$$
> $$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/40\mathbb{Z}$$
> $$\mathbb{Z}/200\mathbb{Z}$$
> Among these abelian groups, the ones containing an element of order 50 are
> $$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$$
> $$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$$
> $$\mathbb{Z}/200\mathbb{Z}$$

5. **True/False.** Prove the following statements if they are correct and disprove if they are wrong.

(a) [**3 points**] If a finite group G acts on a set X in such a way that there is just one orbit for this action, then $|X| \leq |G|$.

> Solution: **True.** Since $X = \cup Gx$ and since there is only one orbit, we have $Gx = X$ for all $x \in X$. By the Orbit-Stabilizer theorem
> $$|X| = |Gx| = \frac{|G|}{|G_x|}$$
> hence $|G| = |G_x||X|$ so $|X| \leq |G|$.

(b) [**3 points**] A group with order 34 has an element with order above 18, then the group is a cyclic group (i.e., the group is generated by one element).

Solution: **True.** Let G be a group with $\#G = 34$. Since the order of an element divides the cardinality of the group and since the divisors of 34 are $1, 2, 17, 34$, if there is an element $g \in G$ of order > 17 then this means that the element g has order 34. Hence $G = \langle g \rangle$.

(c) [**3 points**] Every group of order 99 contains a normal subgroup of order 9.

Solution: **True.** Let G be a group of order 99. By the Sylow theory (Theorem VI.4.3), there is a Sylow 3-group in G of order 9, call this group H. Let $n_3(G)$ denote the number of Sylow 3-groups in G. Then by the same theorem, we have $n_3(G) \equiv 1 \mod 3$ and $n_3(G) \mid 11$ hence $n_3(G) = 1$. Moreover, the same theorem says that the Sylow 3-groups are conjugate. Since we have only one Sylow 3-group, we have $gHg^{-1} = H$ for all $g \in G$. Therefore, the subgroup H is normal hence the statement is correct.